Solutions to the WIKR-06 exam of 21 June 2018

June 14, 2018

1a

We have for $a, b \geq c$:

$$\begin{aligned}
\mathbb{P}(X > a \text{ and } Y > b) &= \int_a^\infty \int_b^\infty \frac{1}{(x+y)^3} \mathrm{d}y \mathrm{d}x = \int_a^\infty \left[-\frac{1}{2(x+y)^2} \right]_b^\infty \mathrm{d}x = \int_a^\infty \frac{1}{2(x+b)^2} \mathrm{d}x \\
&= \left[-\frac{1}{2(x+b)} \right]_a^\infty = \frac{1}{2(a+b)}.
\end{aligned}$$

1b

If $x < c$ then $\mathbb{P}(X \leq x) = 0$ because $f(x,y) = 0$ when $x < c$.

If $x \geq c$ then:

$$F_X(x) = \mathbb{P}(X \leq x) = 1 - \mathbb{P}(X > x) = 1 - \mathbb{P}(X > x \text{ and } Y > c) = 1 - \frac{1}{2(x+c)},$$

where we've used (a) for the last equality.

1c

Because of the form of the joint pdf we must have $\mathbb{P}(X > c \text{ and } Y > c) = 1$. So, using part (a):

$$1 = \frac{1}{2(c+c)},$$

In other words $4c = 1$. So $c = \frac{1}{4}$.

1d

X, Y are **dependent**.

Because:

X, Y are indepdendent iff. for all $a, b \in \mathbb{R}$ we have $\mathbb{P}(X \leq a, Y \leq b) = \mathbb{P}(X \leq a)\mathbb{P}(Y \leq b)$.

Note, for $a, b > c$:

$$\begin{aligned}
\mathbb{P}(X \leq a \text{ and } Y \leq b) &= 1 - \mathbb{P}(X > a \text{ or } Y > b) \\
&= 1 - \mathbb{P}(X > a) - \mathbb{P}(Y > b) + \mathbb{P}(X > a \text{ and } Y > b) \\
&= 1 - \tfrac{1}{2(a+c)} - \tfrac{1}{2(c+b)} + \tfrac{1}{2(a+b)}.
\end{aligned}$$

On the other hand:

$$\mathbb{P}(X \le a)\mathbb{P}(Y \le b) = \left(1 - \frac{1}{2(a+c)}\right) \cdot \left(1 - \frac{1}{2(c+b)}\right) = 1 - \frac{1}{2(a+c)} - \frac{1}{2(c+b)} + \frac{1}{4(a+c)(c+b)}.$$

If X, Y are independent, we must have, for all $a, b \ge c$:

$$\frac{1}{2(a+b)} = \frac{1}{4(a+c)(c+b)}.$$

This is equivalent to $a + b = 2(a+c)(c+b)$ for all $a, b \ge c$.

If you had the correct answer in (c) then you can now for instance fill in $a = b = 1$. From $1 + 1 \ne 2(1 + \frac{1}{4})(1 + \frac{1}{4})$ it follows that X, Y cannot be indepdendent.

If you did not manage to find the answer in (c), then you can reason as follows:

Choose $a = b > c$. If X, Y are independent then we must have $2a = 2(a+c)^2$ for every $a > 0$. In other words, $2a^2 + (2 - 4c)a + 2c^2 = 0$ for *every* $a > c$. This is a quadratic equation in a in which the coefficient of a^2 is nonzero. The equation thus has at most 2 solutions, and certainly not infinitely many solutions. Contradiction!

So X, Y must be dependent.

1e

From (b) it follows that the pdf of X is:

$$f_X(x) = \begin{cases} \frac{1}{2(x+c)^2} & \text{if } x \ge c, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore the expectation of X is:

$$\begin{aligned} \mathbb{E}X &= \int_{-\infty}^{\infty} x f_X(x) \mathrm{d}x = \int_c^{\infty} \frac{x}{2(x+c)^2} \mathrm{d}x = \int_c^{\infty} \frac{1}{2(x+c)} - \frac{c}{2(x+c)^2} \mathrm{d}x \\ &= \left[\frac{1}{2}\ln(x+c) + \frac{c}{2(x+c)}\right]_c^{\infty} = \infty \end{aligned}$$

2a

The pdf of X_1 is:

$$f(x) = \begin{cases} (1/\beta)e^{-x/\beta} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

2b

We have

$$F_M(x) = \mathbb{P}(M \le x) = \mathbb{P}(X_1 \le x, \ldots, X_n \le x) = \mathbb{P}(X_1 \le x) \ldots \mathbb{P}(X_n \le x) = (\mathbb{P}(X_1 \le x))^n = \left(1 - e^{-x/\beta}\right)^n.$$

(The second equality follows from the fac that M is a maximum; for the 3rd and 4th equality we used that the X_i-s are i.i.s and in the last equation we filled in the cdf of X_1.)

The pdf is

$$f_M = F_M' = n(1/\beta)e^{-x/\beta}(1 - e^{-x/\beta})^{n-1}.$$

2c

For every $k \in \mathbb{N}$ we have

$$\mathbb{P}(Y_i > k) = \mathbb{P}(\lceil X_i \rceil > k) = \mathbb{P}(X_i > k) = e^{-k/\beta}.$$

Also note $\mathbb{P}(Y_i > 0) = 1 = e^{-\lambda \cdot 0}$. We can write, for every $k \in \mathbb{N}$:

$$\mathbb{P}(Y_i = k) = \mathbb{P}(Y_i > k-1) - \mathbb{P}(Y_i > k) = e^{-(k-1)/\beta} - e^{-k/\beta} = e^{-(k-1)/\beta}(1 - e^{-1/\beta}).$$

This last expression we recognize as the pdf of a geometric distribution with parameter $p = 1 - e^{-1/\beta}$.

2d

Note $M' = \lceil M \rceil$. So, for every $k \in \mathbb{N}$:

$$\mathbb{P}(M' \leq k) = \mathbb{P}(M \leq k) = \left(1 - e^{-k/\beta}\right)^n.$$

(where we've used (b).)

2e

Note the expression from (d) is als correct when $k = 0$. So, for every $k \in \mathbb{N}$

$$\mathbb{P}(M' = k) = \mathbb{P}(M' \leq k) - \mathbb{P}(M' \leq k-1) = (1-e^{-k/\beta})^2 - (1-e^{-(k-1)/\beta})^2 = 2e^{-(k-1)/\beta} - 2e^{-k/\beta} + e^{-2k/\beta} - e^{-2(k-1)/\beta}.$$

2f

At the beginning of the tontine each participant pays a fee of a and after that each year a total amount of b is paid out, until the last participant has died. (So $M' - 1$ times the amount of b is paid). The expected profit is therefore $a \cdot n - b(\mathbb{E}M' - 1)$, where $n = 2, a = 40, b = n = 2$. That is, the expected profit is $82 - 2\mathbb{E}M'$.

The expectation $\mathbb{E}M'$ can be computed as follows:

$$\mathbb{E}M' = \sum_{k=1}^{\infty} k\mathbb{P}(M' = k) = \sum_{k=1}^{\infty} k \cdot \left(2e^{-(k-1)/\beta} - 2e^{-k/\beta} + e^{-2k/\beta} - e^{-2(k-1)/\beta}\right).$$

This sum we can compute in several ways. One way is as follows. Note $e^{-(k-1)/\beta} - e^{-k/\beta} = e^{-(k-1)/\beta}(1 - e^{-1/\beta})$ and $e^{-2(k-1)/\beta} - e^{-2k/\beta} = e^{-2(k-1)/\beta}(1 - e^{-2/\beta})$. We recognize these expressions as the pmf of geometric distributions with parameters $p_1 := 1 - e^{-1/\beta}$ and $p_2 := 1 - e^{-2/\beta}$. If $X_1 \sim \text{geom}(p_1)$ and $X_2 \sim \text{geom}(p_2)$ then we can write:

$$\begin{aligned}
\mathbb{E}M' &= \sum_{k=1}^{\infty} k\left(2\mathbb{P}(X_1 = k) - \mathbb{P}(X_2 = k)\right) = 2\sum_{k=1}^{\infty} k\mathbb{P}(X_1 = k) - \sum_{k=1}^{\infty} k\mathbb{P}(X_2 = k) \\
&= 2\mathbb{E}X_1 - \mathbb{E}X_2 = \frac{2}{p_1} - \frac{1}{p_2} = \frac{2}{1 - e^{-1/\beta}} - \frac{1}{1 - e^{-2/\beta}}.
\end{aligned}$$

Because $1 - e^{-2/\beta} = (1 - e^{-1/\beta})(1 + e^{-1/\beta})$ we can simplify this to:

$$\mathbb{E}M' = \frac{2(1 + e^{-1/\beta}) - 1}{1 - e^{-2/\beta}} = \frac{1 + 2e^{-1/\beta}}{1 - e^{-2/\beta}}.$$

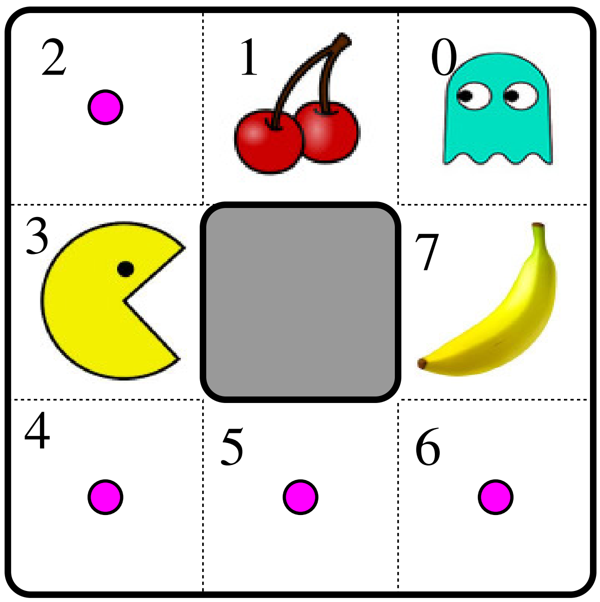
The expected profit is therefore:

$$82 - \frac{2 + 4e^{-1/30}}{1 - e^{-1/15}} \text{ (ducats).}$$

(This turns out to be negative – but estimating the magnitude of this number was not required.)

3a

We number the fields as follows:



The sought system of equations is

$$\begin{aligned} &p_0 = 0, \\ &p_7 = 1, \\ &p_i = \tfrac{1}{2}(p_{i-1} + p_{i+1}), \quad \text{(for } 1 \le i \le 6\text{)}. \end{aligned}$$

We seek p_3

3b

You can recognize the system as the gambler's ruin from the lectures with $p = \frac{1}{2}$ en $N = 7$. You can find a way to solve this system in the lecture notes. We have $p_i = i/7$ and hence the sought probability is $3/7$.

3c

We define:

$A :=$ {Pacman first eats the banana, then the cherry, without coming in contact with the ghost}.
$B :=$ {Pacman eats the banana without coming in contact with either cherry or ghost}

Because $A \subseteq B$, we can compute the probability of A via:

$$\mathbb{P}(A) = \mathbb{P}(A \cap B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \cdot \mathbb{P}(B) = \mathbb{P}(A|B) \cdot \mathbb{P}(B).$$

To compute B we note it corresponds to a standard gambler's ruin with $p = \frac{1}{2}, N = 6$ where the cherry represents bankruptcy and the banana a capital of 6 euro, and pacman starts with a capital of 2 euro. It follows that

$$\mathbb{P}(B) = \frac{2}{6} = \frac{1}{3}.$$

(You can deduce this in the usual way, but quoting the lecture notes is ok as well as long as you state precisely and clearly what you are using.)

To compute the probability of A *given* B we remark that the behavious of Pacman after he has reached the banana is again a (symmetric) random walk, now starting in the square of the banana. The probability of A *given* B is therefore exactly the probability that a symmetric random walk, starting on the square of the banana reaches the cherry without visiting the square with the ghost. In other words, $\mathbb{P}(A|B)$ is again a gambler's ruin with $p = 1/2$ but this time $N = 7$, the ghost represents bankruptcy, the cherry a capital of 7 euro and Pacman starts with a capital of 1 euro. So

$$\mathbb{P}(A|B) = \frac{1}{7}.$$

And therefore

$$\mathbb{P}(A) = \mathbb{P}(A|B) \cdot \mathbb{P}(B) = \frac{1}{7} \cdot \frac{1}{3} = \frac{1}{21}.$$